university of groningen

Exam Computer Assisted Problem Solving (CAPS)

April 1st 2016   9.00-12.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: 10

1. To compute the value of π one could use a numerical method to solve $\sin(x) = 0$, with initial value $x_0 = 3$.

   (a) 5 (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Compute the first two iterations (x_1 and x_2) by means of the Newton method.

   (b) 12 When the iterative method $x_{n+1} = x_n + 2\sin(x_n)$ is used, again with $x_0 = 3$, the results for iterations $1000 - 1004$ are given by

n	x_n
1000	3.10426643
1001	3.17890154
1002	3.10430108
1003	3.17886695
1004	3.10433562

   (1) Will this method eventually convergence? Explain why.
   (2) Determine an error estimate for x_{1004}.
   (3) Calculate an improved solution for x_{1004} by means of Steffensen extrapolation.
   (4) The iterative method $x_{n+1} = x_n + \alpha \sin(x_n)$, with $x_0 = 3$ can be optimized to give optimal linear convergence. Determine the optimal value for α.

   (c) 8 Give a complete program (in pseudo code or Matlab-like language) that solves the problem using the Secant method with an accuracy of tol=1E-6.
   Use an appropriate stopping criterion and start-up procedure.
   Your program should be as computationally efficient as possible.

2. Consider the integral
$$I = \int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{4x}}\, dx$$

   (a) 5 (1) Suppose the Midpoint method is used, and one of the sub-intervals with length $\pi^2/8$ is centered around $x = \pi^2/4$. What is the Rectangular area for this segment?
   (2) Give two reasons why the Midpoint method will not give optimal convergence.

   By means of the substitution $u(x) = \frac{\sqrt{x}}{\pi}$, the integral is reformulated as $I = \int_0^1 \pi \sin(\pi\, u)\, du$. With the Midpoint method the following results are obtained

P.T.O.

n	$I(n)$
16	2.00321638
32	2.00080342
64	2.00020081
128	2.00005020

$I(n)$ is the approximation of the integral on a grid with n sub-intervals.

(b) [12] (1) Compute the q-factor and explain that error estimations are allowed.
Then give an error estimate for $I(128)$ based on subsequent $I(n)$ values.
What wil be the error approximately in case of $n = 256$ segments?
(2) Give the error estimate for $I(128)$ that follows from the global error theorem.
(3) Which estimate is the better one for this integral, theoretically? Explain why.
(4) Compute improved solutions $T_2(128)$ and $T_3(128)$ by means of extrapolation.

(c) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Trapezoidal method (no extrapolation).
Use an appropriate error estimate for the stopping criterion.
Your program should be as computationally efficient as possible.

3. Consider the differential equation $y'(x) = -y^2 + (2x + 1)$, with boundary condition $y(0) = 1$.

(a) [7] Compute the solution at $x = 1.0$:
(1) with Heun's method (RK2) on a grid with $\Delta x = 0.5$.
(2) with the implicit(!) Euler method on a grid with $\Delta x = 1.0$.

With a 2nd order method the solution is determined on 3 grids with $\Delta x = 0.5$, $\Delta x = 0.25$ and $\Delta x = 0.125$. The result at a selection of x locations is as follows:

x_n	$\Delta x = 0.5$	$\Delta x = 0.25$	$\Delta x = 0.125$
0.0	1.00000000	1.00000000	1.00000000
⋮	⋮	⋮	⋮
3.0	2.50715813	2.56470132	2.56793800
4.0	2.64683517	2.93866790	2.94122943
5.0	0.88469351	3.26680831	3.26925026
5.5	-2.54104481	3.41836390	3.42088941
6.0	2.08144535	3.56311215	3.56581974

(b) [9] (1) Give an error estimate for the solution at $x = 4.0$ on the fine grid.
(2) Which grid is approximately required for an error of 1.0E-8?
(3) Give an improved solution (extrapolation) for the solution at $x = 4.0$.

(c) [4] Compute the q-factors for $x = 4$ and $x = 5$ and explain the situation.

(d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the RK2 method (without extrapolation).
Use an appropriate error estimate for the stopping criterion.

4. Consider the diff. eqn. $y''(x) + \alpha\, y'(x) + 2\, y(x) = \cos(\pi\, x)$, with boundary conditions $y(0) = 1$ and $y(1) = 0$.

(a) [7] Take $\alpha = 0$, such that $y'(x)$ is (temporarily) out of the diff. eqn.
Describe the matrix and rhs-vector when the problem is solved on a grid with $N = 10$ segments by means of the matrix method, using the [1 -2 1]-formula for $y''(x)$.

(b) [5] (1) Which modification do you have to make to the system when $\alpha = 3$?
(2) There are a number of options to put $y'(x)$ in the system. Which option only changes the diagonals L and R, but does not change D?

Total: [100]